The Volatility Surface

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1. A volatility surface is given as a function of maturity and strike. Data provider collect the price for that option and do invert it with Black formula or, when it comes to interest rate option, with the equivalent equation for a log-normal shifted model. Volatility surfaces, suitably interpolated and extended, are then used to compute values of other options or assets with embedded optionality.
2. When computing the amount of capital at risk, banks work under the hypothesis that the dynamic of the underlying interest rate remains unchanged, while the initial condition, namely the indexing curve, is stressed by a fixed amount. This is normally done holding fixed the volatility surface associated to the observed indexing curve.

Indice

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The Forward Price

Let $X(t)$ the price process of an asset for which we have a forward market, the spot market may or may not exist so $X(t)/B(t)$ needs not to be a martingale with respect to the risk neutral process. The forward value F_T is the price, agreed today, that I will pay in T for that asset. The price is determined in such a way that today's value of the contract is zero. Based on these assumption the forward price turns out to be the solution of the equation:

$$\mathbb{E}\left[\frac{X(T) - F_T}{B(T)}\right] = 0$$

that is:

$$\begin{aligned} P(0,T)\, F_T &= B(0)\,\mathbb{E}\left[\frac{X(T)}{B(T)}\right]. \\ F_T &= \mathbb{E}^{P_T}[\, X(T)\,] \end{aligned}$$

The Forward Price Process

The natural extension of the forward price, is the 'forward price process'. The idea of working with forward price processes dates wa back, so nothing is new here, we just review the notation for sake of completeness. We can think of striking contracts, as the one described above, at any time in the future. What that price will be tomorrow is a RV conditional to the market condition at 't' and it will be defined in the same way. The mathematical way to express this concept is to define the forward price process $F(t, T)$ as the solution of the (stochastic equation)

$$\mathbb{E}\left[\frac{X(T) - F(t,T)}{B(T)}\middle|\mathcal{F}_t\right] = 0. \quad (1)$$

It is easy to check that:

$$P(t,T)F(t,T) = \mathbb{E}\left[\frac{B(t)}{B(T)}X(T)\middle|\mathcal{F}_t\right] \quad (2)$$

The Forward Price Process

and performing the usual numeraire change:

$$F(t,T) = \mathbb{E}^{P_T}\left[X(T) \middle| \mathcal{F}_t \right]. \tag{3}$$

An immediate consequence are the relations:

$$F(0,T) = F_T, \quad F(T,T) = X(T). \tag{4}$$

The second equality of this equation is rather important. It says that any time we have a (vanilla) payoff, written for $X(T)$, as the result of the dynamics of the process $X(t)$, the same payoff can be computed replacing $X(t)$ with $F(t,T)$.
If $X(t)/B(t)$ is a martingale with respect to the measure $B(t)$, from eq.(2) we get:

$$F(t,T) = \frac{X(t)}{P(t,T)}, \quad \text{that implies} \quad F_T = \frac{X(0)}{P(0,T)}, \tag{5}$$

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The Forward Price Process

A valid forward price process must be a martingale when the measure is the one induced by the numeraire $P(t,T)$, therefore we will model exactly that property:

- $M(t,T,\theta_{t_0})$ a martingale process with respect to the measure $P(t,T)$
- θ_{t_0} is the set of parameters defining the model at $t=t_0$

We model the forward price process of our asset as:

$$F(t,T) = F_T M(t,T,\theta_{t_0}), \quad M(0,T,\theta_{t_0}) = 1, \quad 0 \le t \le T. \tag{6}$$

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Positive Martingale

On the market, for each strike κ and maturity T we observe a price $\Pi(\kappa, F_T)$ that depends (at least in principle) upon the forward value of the underlying.
Since we are true to our art, we believe in non arbitrage markets and mathematical theorems, so we must admit that there exists a stochastic process describing the dynamics of the forward process $F(t, T)$ and producing a martingale measure that can reproduce all of the observed prices.

Positive Martingale

Let $\Pi(\kappa, F_T)$ the observed (put) forward option price on the market, we know that we must have

$$\begin{aligned}
\Pi(\kappa, F_T) &= \kappa \mathbb{E}^{P_T}\left(\mathbf{1}_{[M(T,T,\boldsymbol{\theta}_{t_0})<\rho]}\right) \\
&- F_T \mathbb{E}^{P_T}\left(M(T,T,\boldsymbol{\theta}_{t_0})\mathbf{1}_{[M(T,T,\boldsymbol{\theta}_{t_0})<\rho]}\right), \\
&\rho := \frac{\kappa}{F_T}.
\end{aligned} \tag{7}$$

Positive Martingale

If the martingale measure is generated by a positive pocess we can make some significant progress. In this case we can write:

$$\Pi(\kappa, F_T) = \kappa \mathbb{P}^{P_T}\left(M(T,\boldsymbol{\theta}) < \rho\right) - F_T \mathbb{P}^{M_T}\left(M(T,\boldsymbol{\theta}) < \rho\right), \tag{8}$$

where $\mathbb{P}^{P_T}$ is the distribution function with respect to the $P(t, T)$ measure, while $\mathbb{P}^{M_T}$ is the distribution function with respect to the measure were we have selected $M(t, T, \boldsymbol{\theta}_{t_0})$ as numeraire.
If we divide both sides of eq.(8) by F_T we see that:

$$\frac{\Pi(\kappa, F_T)}{F_T} = G(T, \rho, \boldsymbol{\theta}_{t_0}), \tag{9}$$

where G is a function that, besides the dynamics imposed by $\boldsymbol{\theta}$ and the maturity T, depends solely on the homogeneous parameter ρ.

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The Volatility Surface

Whatever is the theory $M(t, T, \boldsymbol{\theta}_{t_0})$ from eq.(8) we see that the price $\Pi(\kappa, F_T)$ must respect the non arbitrage bounds

$$\max((\kappa - F_T), 0) \leq \Pi(\kappa, F_T) \leq \kappa. \tag{10}$$

The upper bound is obvious, while the lower one comes from Jensen's inequality and the convex shape of the payoff. In virtue of eq.(10) we can find a value of σ such that the Black-Scholes martingale $M(t, \sigma)$ will produce the correct price:

$$\Pi(\kappa, F_T) = \kappa \mathbb{P}^{P_T}\left(M(T, \sigma) < \rho\right) - F_T \mathbb{P}^{M_T}\left(M(T, \sigma) < \rho\right). \tag{11}$$

The function associating to each pair (κ, F_T) the correct value of $\sigma(\kappa, F_T)$ defines the implied volatility surface.

The Volatility Surface

The homogeneous properties displayed by eq.(9) are true for the BS martingale as well and we have:

$$\frac{\Pi(\kappa, F_T)}{F_T} = \rho \mathbb{P}^{P_T}\left(M(T,\sigma) < \rho\right) - \mathbb{P}^{M_T}\left(M(T,\sigma) < \rho\right) = H(T,\rho,\sigma). \tag{12}$$

Comparison of eq.(9) with eq.(12) will require that

$$G(T,\rho,\boldsymbol{\theta}_{t_0}) = H(T,\rho,\sigma), \tag{13}$$

with the obvious consequence that we obtain the scaling equation:

$$\sigma = \mathcal{S}(T,\rho,\boldsymbol{\theta}_{t_0}). \tag{14}$$

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Numerical Results

Although eq.(14) is pretty easy to obtain and, in our opinion rather solid, it is still worth providing numerical evidence of its validity. We have computed prices of vanilla options for both the Heston model and the Variance Gamma model, and we have extracted the implied volatility smile. This has been done for three different values of underlying $S_o = 0.9, 1.0, 1.1$ and four distinct maturities, 15 days, 1 month, 6 months and 1 year. The value of the implied volatility has been plotted versus the homogeneous parameter $\rho = \kappa / F_T$.
The same surface has been plotted versus the parameter $\kappa - F_T$. It is pretty cleat that the hypothesis that the volatility surface depends on $\kappa - F_T$ (sticky strike) is unteneable.

Numerical Results

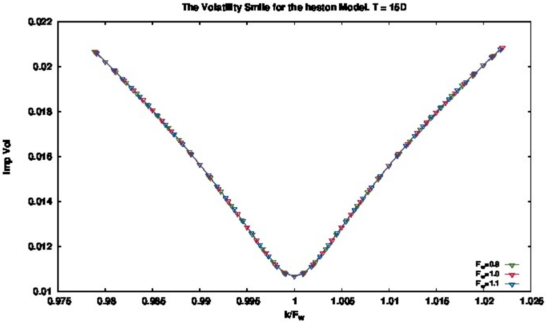


Figure : The volatility smile of the Heston model for different values of the underlying. The maturity is 15 days, model parameters have been selected to stress the 'smile' effect. In the x-axis we have the homogeneous forward moneyness.

Numerical Results

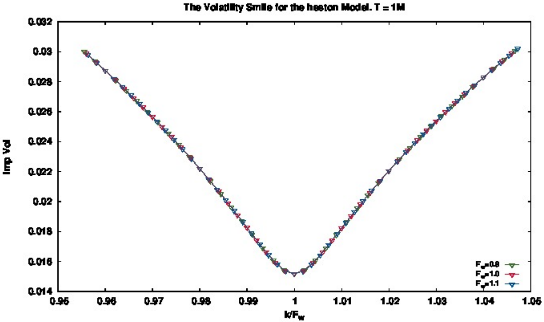


Figure : The volatility smile of the Heston model for different values of the underlying. The maturity is one month, model parameters have been selected to stress the 'smile' effect. In the x-axis we have the homogeneous forward moneyness.

Numerical Results

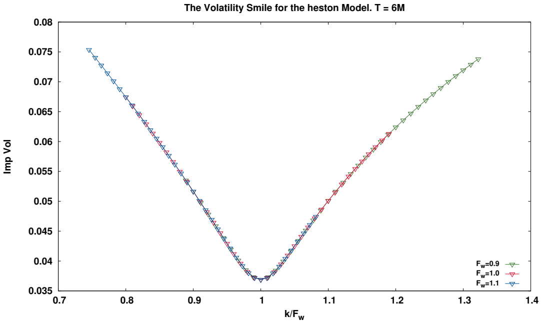


Figure : The volatility smile of the Heston model for different values of the underlying. The maturity is six months, model parameters have been selected to stress the 'smile' effect. In the x-axis we have the homogeneous forward moneyness.

Numerical Results

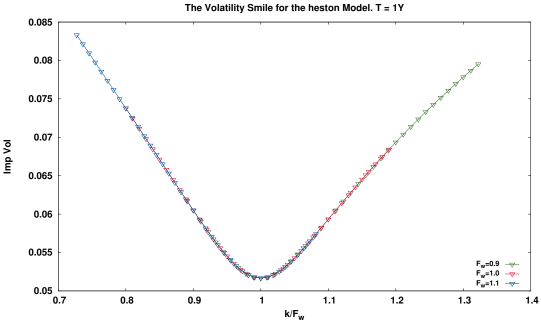


Figure : The volatility smile of the Heston model for different values of the underlying. The maturity is one year, model parameters have been selected to stress the 'smile' effect. In the x-axis we have the homogeneous forward moneyness.

Numerical Results

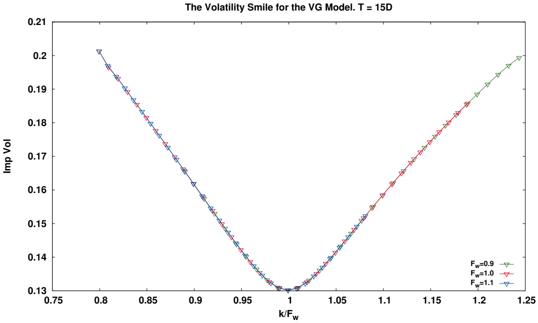


Figure : The volatility smile of the Variance-Gamma model for different values of the underlying. The maturity is fifteen days, model parameters have been selected to stress the 'smile' effect. In the x-axis we have the homogeneous forward moneyness.

Numerical Results

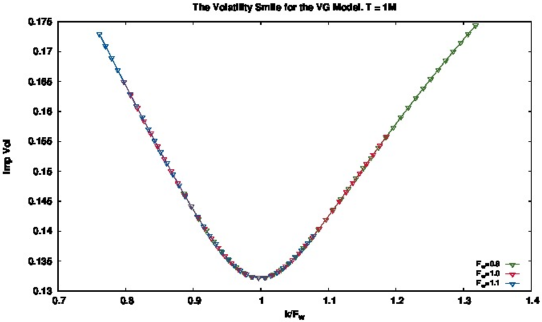


Figure : The volatility smile of the Variance-Gamma model for different values of the underlying. The maturity is one month, model parameters have been selected to stress the 'smile' effect. In the x-axis we have the homogeneous forward moneyness.

Numerical Results

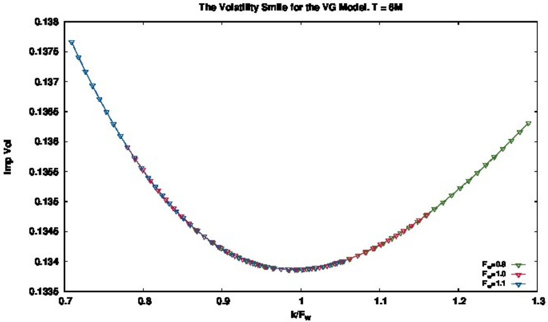
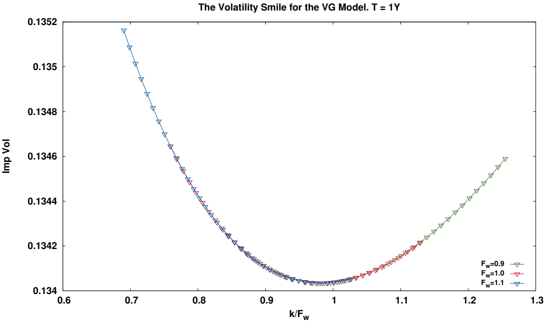


Figure : The volatility smile of the Variance-Gamma model for different values of the underlying. The maturity is six months, model parameters have been selected to stress the 'smile' effect. In the x-axis we have the homogeneous forward moneyness.

Numerical Results



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Numerical Results

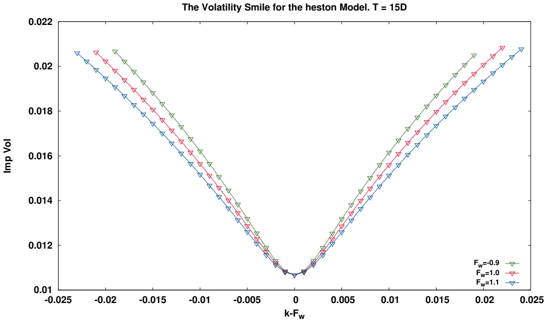
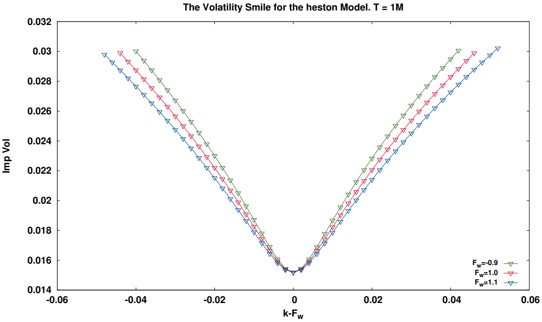


Figure : The volatility smile of the Heston model for different values of the underlying. The maturity is 15 days, model parameters have been selected to stress the 'smile' effect. In the x-axis we have $K - F_w$.

Numerical Results



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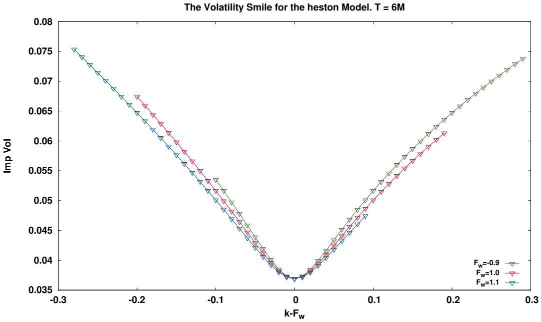

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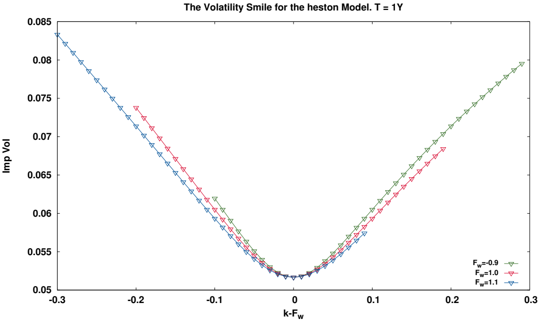

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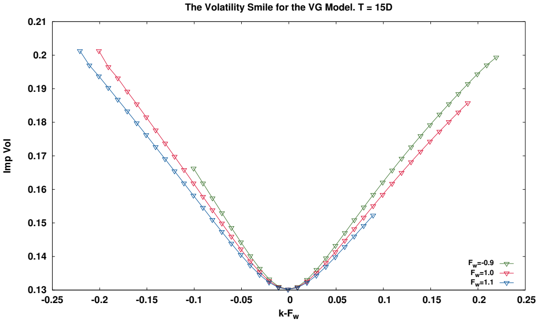
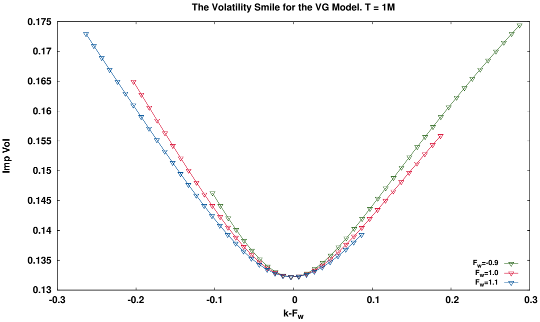


Figure : The volatility smile of the Variance-Gamma model for different values of the underlying. The maturity is fifteen days, model parameters have been selected to stress the 'smile' effect. In the x-axis we have $K - F_w$.

Numerical Results



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Numerical Results

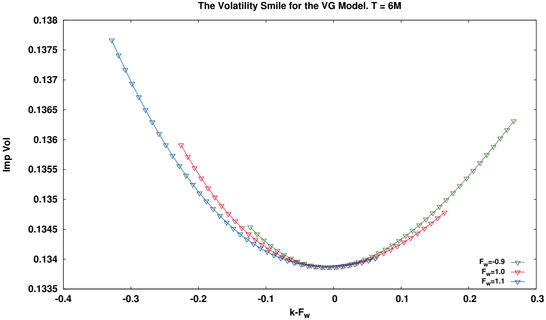

Figure : The volatility smile of the Variance-Gamma model for different values of the underlying. The maturity is one month, model parameters have been selected to stress the 'smile' effect. In the x-axis we have $K - F_w$.

Numerical Results

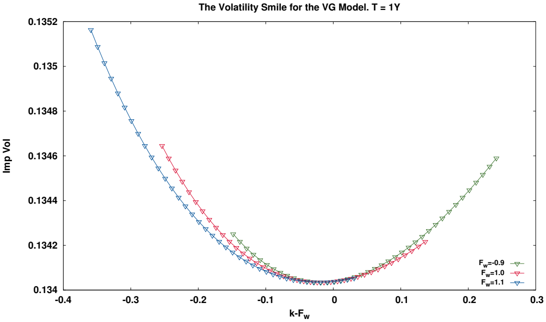

Figure : The volatility smile of the Variance-Gamma model for different values of the underlying. The maturity is one year, model parameters have been selected to stress the 'smile' effect. In the x-axis we have $K - F_w$.

Applications

There are at least two important applications of the scaling equation (13) both associated to risk management. One is related to stressed scenarios, while the other deals with proper delta-hedging of positions exposed to market risk.
In a stressed scenarios situation we would like to compute the price of the option under the assumption that the forward rate has been stressed to a new value F'_T, all else being equal. In the language of the previous section we would say that the new model for the forward process will be:

$$F(t, T) = F'_T M(t, \theta).$$

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Applications

The usual way to do so is to use again the BS martingale with the new scenario. The problem is that we do not know the prices in this 'stressed' world, the world F'_T is not observable, we only know about F_T and, as a consequence we do not know the implied volatility to be used for the situation. We can estimate the value to use from the scaling equation (14). In fact if we determine a value κ' such that

$$\frac{\kappa'}{F_T} = \frac{\kappa}{F'_T},$$

we have immediately that:

$$\sigma(\kappa, F'_T) = \mathcal{S}\left(\frac{\kappa}{F'_T}, \boldsymbol{\theta}\right) = \mathcal{S}\left(\frac{\kappa'}{F_T}, \boldsymbol{\theta}\right) = \sigma(\kappa', F_T),$$

and this value can be read off the known volatility surface.

Applications

A proper Δ-hedging strategy requires the computation of the Δ sensitivity. This one is given by:

$$\Delta := \frac{d}{dF_T}\Pi(\kappa, F_T) = \frac{\partial}{\partial F_T}\Pi(\kappa, F_T) + \frac{\partial}{\partial \sigma}\Pi(\kappa, F_T)\frac{\partial \sigma}{\partial F_T}. \qquad (15)$$

The first term of the last expression, the Black-Scholes sensitivity Δ_{BS}, is given by

$$\frac{\partial}{\partial F_T}\Pi(\kappa, F_T) = -\mathbb{P}^{M_T}(M(T,\sigma) < \rho),$$

and corresponds to what we would have computed if we had left unchanged the surface. The quantity

$$\frac{\partial}{\partial \sigma}\Pi(\kappa, F_T)$$

is the Vega ($\mathcal{V}$) from the Black-Scholes model and is a strictly positive term.

The scaling equation (14) says that the last term in eq.(15) can be written as:

$$\frac{d\sigma}{dF_T} = -\rho\frac{d\sigma}{d\kappa},$$

and we can write:

$$\Delta = \Delta_{BS} - \rho\mathcal{V}\frac{d\sigma}{d\kappa}. \tag{16}$$

From this equation we see that the Δ_{BS} (a negative term) is, in absolute value, an underestimate of the *true* Δ when the smile is slanted upward, while it is an overestimate when the smile is slanted downward.

Interest Rates

In the previous discussion we have freely mentioned forward processes without making any distinction whether we were dealing with assets processes or intrest rate processes. When it comes to market models of interest rates, the situation is slightly different. The existence of negative rates rules out the possibility that we can relay on a positive martingale to realize the observed prices.
We have, at least, to extend the model to an affine transformation of a positive martingale and we model the forward rate as:

$$F(t,T) = (F_T + \lambda_\theta) M(t, \boldsymbol{\theta}) - \lambda_\theta$$

In this situation the price of the put option is given by:

$$\Pi(\kappa, F_T) = (\kappa + \lambda_\theta)\mathbb{P}^{P_T}(M(T, \boldsymbol{\theta}) < \rho(\lambda_\theta)) - (F_T + \lambda_\theta)\mathbb{P}^{M_T}(M(T, \boldsymbol{\theta}) < \rho(\lambda_\theta)),$$

where

$$\rho(\lambda_\theta) = \frac{\kappa + \lambda_\theta}{F_T + \lambda_\theta}.$$

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Interest Rates

We would like to find a σ such that a shifted BS martingale

$$(F_T + \lambda_\sigma)M(t, \sigma) - \lambda_\sigma$$

could reproduce the observed price. But, just looking at the upper bound for the put option, that is $\kappa + \lambda$ we can immediately see that such a σ is guaranteed to exists only if we select $\lambda_\sigma \geq \lambda_\theta$. So, let's assume that the inequality holds strictly. In this case we have still some problem, given that the equivalent of eq.(13) would be:

$$G(\rho(\lambda_\theta), \boldsymbol{\theta}) = H(\rho(\lambda_\sigma), \sigma) \tag{17}$$

forcing the conclusion that

$$\sigma = \mathcal{S}(\rho(\lambda_\theta), \rho(\lambda_\sigma), \boldsymbol{\theta}). \tag{18}$$

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In dealing with negative rates we are forced to assume that $\lambda_\theta = \lambda_\sigma$ (this could be done calibrating on the $\lambda_\sigma$parameter) and all of the above results will hold after replacing ρ with $\rho(\lambda)$.
As an example we produced in a volatitily surface for a shifted log-normal model while the θ martingale was a shifted Heston model. The difference between the two is that in we had $\lambda_\theta = 150$ *bps* and $\lambda_\sigma = 180$ *bps*, and a second one where both λ_θ and λ_σ were chosen at 150*bps*. The full agreement with eq.(18) is obvious.

Interest Rates

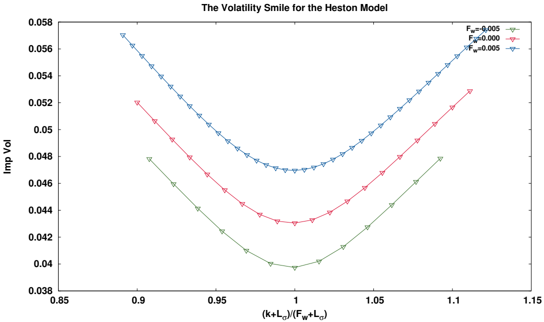

Figure : The volatility smile of the Heston model for different values of the underlying. The maturity is one year, model parameters have been selected to stress the 'smile' effect. In the x-axis we have the homogeneous forward moneyness. In this plot we have $\lambda_\theta = 150 bps$ while $\lambda_\sigma = 180 bps$.

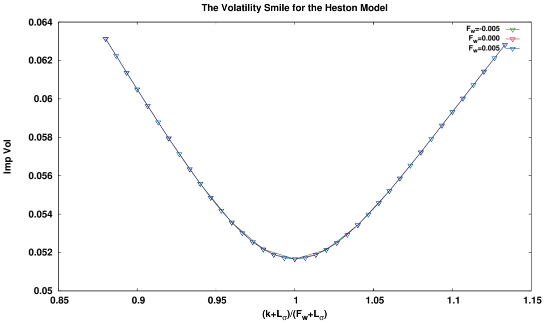

Figure : The volatility smile of the heston model for different values of the underlying. The maturity is one year, model parameters have been selected to stress the 'smile' effect. In the x-axis we have the homogeneous forward moneyness. In this splot we have $\lambda_\theta = \lambda_\sigma = 150bps$.

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Market Data

It was hard to resist the temptation to check this simple theoretical result with market data. If we confine to martingales with only constant parameters, we should observe the scaling law when we look at surfaces produced in different days.
The equation becomes a 'universal' equation:

$$G(T, \rho, \boldsymbol{\theta}) = H(T, \rho, \sigma),$$

We looked at the volatility surface of the Euro Stoxx 50 in three different days. May 31^{st}, June 30^{th} and July 31^{st} of 2017. We did concentrate on 1-month and 1 -year maturity options and we tried to check for the parametrisation law derived for theoretical models.

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Market Data

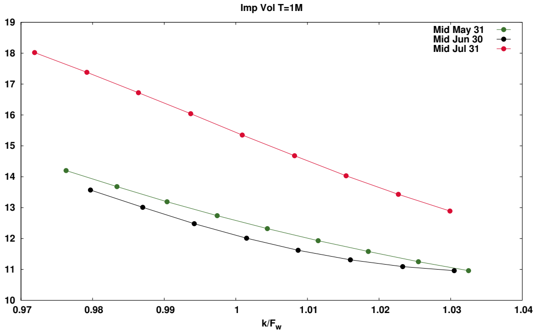
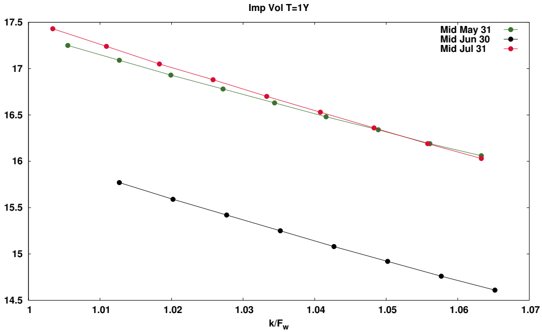


Figure : The volatility smile of the Euro Stoxx 50 for three different days. The maturity is one month. In the x-axis we have the homogeneous forward moneyness κ/F_T.

Market Data



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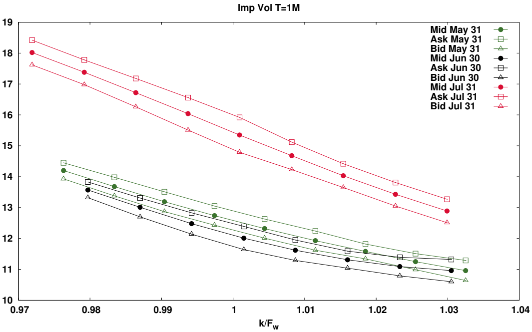

Figure : The volatility smile of the Euro Stoxx 50 for three different days. The maturity is one month. In the x-axis we have the homogeneous forward moneyness κ/F_T. For data relative to May 31st we have plotted the smile obtained by the bid-ask spread.

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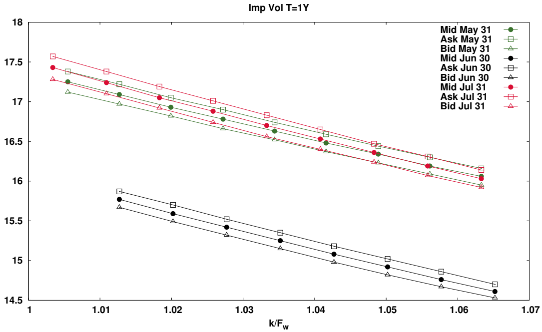
Market Data



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Conclusions

As we have seen in the more theoretical sections, to produce risk measures and sensitivity coherent with an underlying martingale measure, it is necessary to include the dependency of the volatility surface.

Unfortunately, coherence with a theoretical framework is not enough to assure coherence with the real world. In our view the results on the Euro Stoxx 50 rule out the possibility that the underlying martingale measure is produced by a positive martingale with constant coefficients. We believe that there are no painless ways out of this riddle. Even if we deal with a process that, in the real world, attains only positive values, nothing says that we have to model it with a positive martingale.

Conclusions

The trick of shifting a positive martingale leaves us with a sour taste, given that, in the shifted framework turns out to be difficult to decouple the dynamics from the initial conditions. A shift must be tuned to the value of the observed forward, and that would inevitably meddle with the dynamics. If we resort to more general non positive martingales the price bound described in eq. (10) will not hold anymore and, as a consequence, will break the one to one mapping between prices and implied volatility. This may or may not be a theoretical problem, but surely would make the model testing much harder.
To relay on hidden variables (stochastic volatility for example) seems to be the 'sensible' way to go.