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ARPM | Advanced Risk and Portfolio Management®

Black-Litterman and Beyond

Quantitative Finance @ WORK
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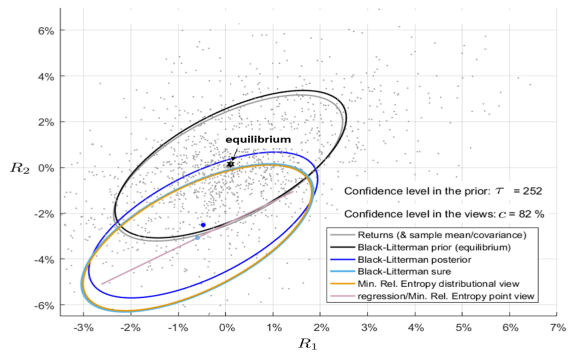
The Black-Litterman approach has received much attention in the investment management community as the reference methodology to process subjective views on the markets, in a way that gives rise to sensible portfolios. However, the application of the Black-Litterman approach is not always straightforward, as multiple steps are involved to implement the approach.

Furthermore, the Black-Litterman approach is ideal for views on expected returns in normal markets, but not suitable for more general settings, such as views on correlations and volatilities and/or views in non-normal markets.

In this presentation we walk through the separate conceptual steps of the Black-Litterman approach. Then we discuss a theoretical framework based on relative entropy minimization that allows us to process arbitrary views on markets with arbitrary distributions. Third, we discuss implementations of the above theoretical framework

Black-Litterman versus entropy

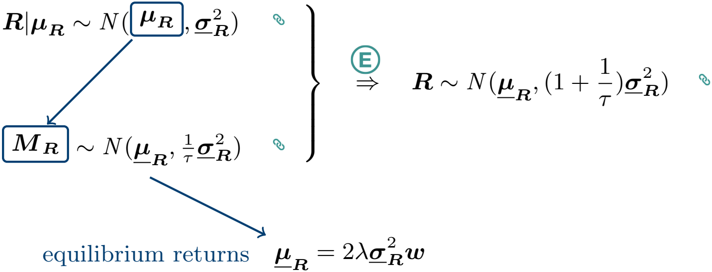


- number of realizations: $\bar{t} = 1,599$
- number of instruments: $\bar{n} = 2$
- τ (= confidence level in the prior) and c (= confidence level in the views) vary

Equilibrium prior distribution

$$\begin{tikzcd}
\boldsymbol{R}|\boldsymbol{\mu}_{\boldsymbol{R}} \sim N(\boxed{\boldsymbol{\mu}_{\boldsymbol{R}}}, \underline{\boldsymbol{\sigma}}^2_{\boldsymbol{R}}) \arrow[d] \\
\boxed{\boldsymbol{M}_{\boldsymbol{R}}} \sim N(\underline{\boldsymbol{\mu}}_{\boldsymbol{R}}, \tfrac{1}{\tau}\underline{\boldsymbol{\sigma}}^2_{\boldsymbol{R}})
\end{tikzcd}
\left.\vphantom{\begin{matrix}a\\a\\a\\a\end{matrix}}\right\} \overset{\text{Ⓔ}}{\Rightarrow} \quad \boldsymbol{R} \sim N(\underline{\boldsymbol{\mu}}_{\boldsymbol{R}}, (1+\frac{1}{\tau})\underline{\boldsymbol{\sigma}}^2_{\boldsymbol{R}})$$

Equilibrium prior distribution



$$\left.\begin{array}{l} \boldsymbol{R}|\boldsymbol{\mu}_{\boldsymbol{R}} \sim N(\boxed{\boldsymbol{\mu}_{\boldsymbol{R}}}, \underline{\boldsymbol{\sigma}}^2_{\boldsymbol{R}}) \\ \\ \boxed{\boldsymbol{M}_{\boldsymbol{R}}} \sim N(\underline{\boldsymbol{\mu}}_{\boldsymbol{R}}, \tfrac{1}{\tau}\underline{\boldsymbol{\sigma}}^2_{\boldsymbol{R}}) \end{array}\right\} \overset{\textcircled{E}}{\Rightarrow} \quad \boldsymbol{R} \sim N(\underline{\boldsymbol{\mu}}_{\boldsymbol{R}}, (1+\frac{1}{\tau})\underline{\boldsymbol{\sigma}}^2_{\boldsymbol{R}})$$

equilibrium returns $\underline{\boldsymbol{\mu}}_{\boldsymbol{R}} = 2\lambda\underline{\boldsymbol{\sigma}}^2_{\boldsymbol{R}}\boldsymbol{w}$

$$\Updownarrow$$

market portfolio weights $\boldsymbol{w} = \dfrac{1}{2\lambda}(\underline{\boldsymbol{\sigma}}^2_{\boldsymbol{R}})^{-1}\underline{\boldsymbol{\mu}}_{\boldsymbol{R}}$

Three stocks from the S&P 500

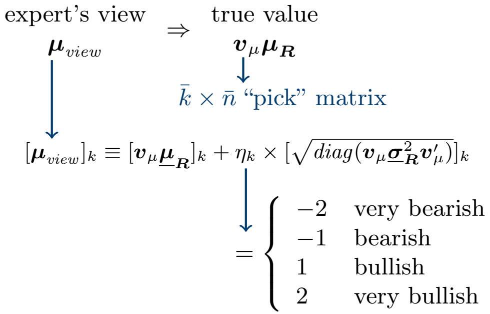
Black-Litterman linear views

$$\begin{array}{c} \text{expert's view} \\ \boldsymbol{\mu}_{view} \end{array} \Rightarrow \begin{array}{c} \text{true value} \\ \boldsymbol{v}_{\mu}\boldsymbol{\mu}_{\boldsymbol{R}} \end{array}$$

Black-Litterman linear views

$$\begin{array}{ccc}\text{expert's view} & \Rightarrow & \text{true value}\\ \boldsymbol{\mu}_{view} & & \underset{\substack{\downarrow\\ \bar{k}\times\bar{n}\text{ “pick” matrix}}}{\boldsymbol{v}_{\mu}}\boldsymbol{\mu}_{\boldsymbol{R}}\end{array}$$

Black-Litterman linear views



expert's view ⇒ true value

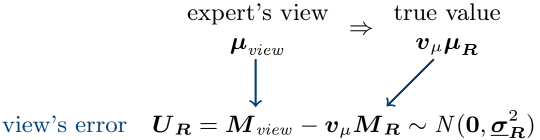
$\boldsymbol{\mu}_{view}$ $\qquad$ $\boldsymbol{v}_{\mu}\boldsymbol{\mu}_{\boldsymbol{R}}$

$\bar{k} \times \bar{n}$ "pick" matrix

$$[\boldsymbol{\mu}_{view}]_k \equiv [\boldsymbol{v}_{\mu}\underline{\boldsymbol{\mu}}_{\boldsymbol{R}}]_k + \eta_k \times [\sqrt{diag(\boldsymbol{v}_{\mu}\underline{\boldsymbol{\sigma}}^2_{\boldsymbol{R}}\boldsymbol{v}'_{\mu})}]_k$$

$$= \begin{cases} -2 & \text{very bearish} \\ -1 & \text{bearish} \\ 1 & \text{bullish} \\ 2 & \text{very bullish} \end{cases}$$

Black-Litterman linear views



expert's view $\Rightarrow$ true value

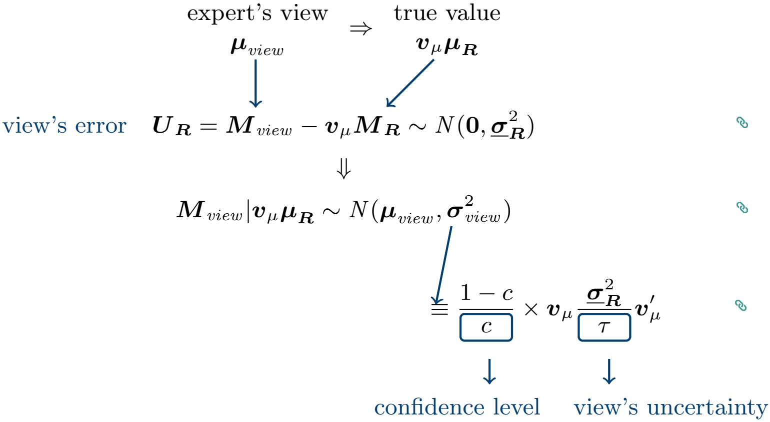
$\boldsymbol{\mu}_{view}$ $\quad$ $\boldsymbol{v}_{\mu}\boldsymbol{\mu}_{\boldsymbol{R}}$

view's error $\quad \boldsymbol{U}_{\boldsymbol{R}} = \boldsymbol{M}_{view} - \boldsymbol{v}_{\mu}\boldsymbol{M}_{\boldsymbol{R}} \sim N(\boldsymbol{0}, \underline{\boldsymbol{\sigma}}^2_{\boldsymbol{R}})$

$$\Downarrow$$

$$\boldsymbol{M}_{view}|\boldsymbol{v}_{\mu}\boldsymbol{\mu}_{\boldsymbol{R}} \sim N(\boldsymbol{\mu}_{view}, \boldsymbol{\sigma}^2_{view})$$

Black-Litterman linear views



expert's view ⇒ true value

$\boldsymbol{\mu}_{view}$ $\boldsymbol{v}_{\mu}\boldsymbol{\mu}_{\boldsymbol{R}}$

view's error $\boldsymbol{U}_{\boldsymbol{R}} = \boldsymbol{M}_{view} - \boldsymbol{v}_{\mu}\boldsymbol{M}_{\boldsymbol{R}} \sim N(\boldsymbol{0}, \underline{\boldsymbol{\sigma}}^{2}_{\boldsymbol{R}})$

$$\Downarrow$$

$$\boldsymbol{M}_{view}|\boldsymbol{v}_{\mu}\boldsymbol{\mu}_{\boldsymbol{R}} \sim N(\boldsymbol{\mu}_{view}, \boldsymbol{\sigma}^{2}_{view})$$

$$\equiv \frac{1-c}{\boxed{c}} \times \boldsymbol{v}_{\mu}\frac{\underline{\boldsymbol{\sigma}}^{2}_{\boldsymbol{R}}}{\boxed{\tau}}\boldsymbol{v}'_{\mu}$$

confidence level view's uncertainty

💡 **Three stocks from the S&P 500**

Sanity check

Correlation matrix

$$corr(\boldsymbol{v}_{\mu}\underline{\boldsymbol{\sigma}}^{2}\boldsymbol{v}_{\mu}')$$

$$\Downarrow$$

Normalized eigenvalues

$$w_n \equiv \frac{\lambda_n^2}{\sum_{m=1}^{\bar{n}} \lambda_m^2}$$

$$\Downarrow$$

Effective rank

$$\mathit{eff\_rank}(corr(\boldsymbol{v}_{\mu}\underline{\boldsymbol{\sigma}}^{2}\boldsymbol{v}_{\mu}')) \equiv e^{-\sum w_n \log w_n}$$

$$\Downarrow$$

Sanity condition

$$\mathit{eff\_rank}(corr(\boldsymbol{v}_{\mu}\underline{\boldsymbol{\sigma}}^{2}\boldsymbol{v}_{\mu}')) \geq \alpha$$

Three stocks from the S&P 500

Posterior distribution

$$\boldsymbol{R}|\boldsymbol{\mu}^{view} = \boldsymbol{M}_{\boldsymbol{R}}|\boldsymbol{\mu}^{view} + \boldsymbol{U}_{\boldsymbol{R}}$$

$$\Downarrow$$

$$\boldsymbol{R}|\boldsymbol{\mu}^{view} \sim \bar{f}_{\boldsymbol{R}} \sim N(\bar{\boldsymbol{\mu}}^{BL}_{\boldsymbol{R}}, \bar{\boldsymbol{\sigma}}^{2BL}_{\boldsymbol{R}})$$

$$= \mathbb{E}\{\boldsymbol{M}_{\boldsymbol{R}}|\boldsymbol{\mu}^{view}\} = \underline{\boldsymbol{\mu}}_{\boldsymbol{R}} + \frac{1}{\tau}\underline{\boldsymbol{\sigma}}^{2}_{\boldsymbol{R}}\boldsymbol{v}'_{\mu}(\frac{1}{\tau}\boldsymbol{v}_{\mu}\underline{\boldsymbol{\sigma}}^{2}_{\boldsymbol{R}}\boldsymbol{v}'_{\mu} + \boldsymbol{\sigma}^{2view})^{-1}(\boldsymbol{\mu}^{view} - \boldsymbol{v}_{\mu}\underline{\boldsymbol{\mu}}_{\boldsymbol{R}})$$

,

Posterior distribution

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$$\boldsymbol{R}|\boldsymbol{\mu}^{view} \sim \bar{f}_{\boldsymbol{R}} \sim N(\bar{\boldsymbol{\mu}}^{BL}_{\boldsymbol{R}}, \bar{\boldsymbol{\sigma}}^{2BL}_{\boldsymbol{R}})$$

Ⓔ

$$= \mathbb{C}v\{\boldsymbol{M}_{\boldsymbol{R}}|\boldsymbol{\mu}^{view}\} + \mathbb{C}v\{\boldsymbol{U}_{\boldsymbol{R}}\} = \bar{\boldsymbol{\psi}}^{2} + \underline{\boldsymbol{\sigma}}^{2}_{\boldsymbol{R}}$$

Posterior distribution

$$\boldsymbol{R}|\boldsymbol{\mu}^{view} = \boldsymbol{M}_{\boldsymbol{R}}|\boldsymbol{\mu}^{view} + \boldsymbol{U}_{\boldsymbol{R}}$$

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Ⓔ

$$= \mathbb{C}v\{\boldsymbol{M}_{\boldsymbol{R}}|\boldsymbol{\mu}^{view}\} + \mathbb{C}v\{\boldsymbol{U}_{\boldsymbol{R}}\} = \bar{\boldsymbol{\psi}}^2 + \underline{\boldsymbol{\sigma}}^2_{\boldsymbol{R}}$$

Ⓔ

$$= ((\frac{1}{\tau}\underline{\boldsymbol{\sigma}}^2_{\boldsymbol{R}})^{-1} + \boldsymbol{v}'_{\mu}(\boldsymbol{\sigma}^{2view})^{-1}\boldsymbol{v}_{\mu})^{-1}$$

$$\Downarrow$$

$$\bar{\boldsymbol{\sigma}}^{2BL}_{\boldsymbol{R}} \equiv (1 + \frac{1}{\tau})\underline{\boldsymbol{\sigma}}^2_{\boldsymbol{R}} - \frac{1}{\tau}\underline{\boldsymbol{\sigma}}^2_{\boldsymbol{R}}\boldsymbol{v}'_{\mu}(\boldsymbol{v}_{\mu}\underline{\boldsymbol{\sigma}}^2_{\boldsymbol{R}}\boldsymbol{v}'_{\mu})^{-1}\boldsymbol{v}_{\mu}\underline{\boldsymbol{\sigma}}^2_{\boldsymbol{R}} \approx \underline{\boldsymbol{\sigma}}^2_{\boldsymbol{R}}$$

Three stocks from the S&P 500

Limit cases

- High confidence in prior

$$\boldsymbol{R} \sim N(\bar{\boldsymbol{\mu}}_{\boldsymbol{R}}^{BL}, \bar{\boldsymbol{\sigma}}_{\boldsymbol{R}}^{2BL}) \quad \underset{\tau \to \infty}{\to} \quad \boldsymbol{R} \sim N(\underline{\boldsymbol{\mu}}_{\boldsymbol{R}}, \underline{\boldsymbol{\sigma}}_{\boldsymbol{R}}^{2})$$

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$$\boldsymbol{R} \sim N(\bar{\boldsymbol{\mu}}_{\boldsymbol{R}}^{BL}, \bar{\boldsymbol{\sigma}}_{\boldsymbol{R}}^{2BL}) \quad \underset{\tau \to \infty}{\to} \quad \boldsymbol{R} \sim N(\underline{\boldsymbol{\mu}}_{\boldsymbol{R}}, \underline{\boldsymbol{\sigma}}_{\boldsymbol{R}}^{2})$$

- Low confidence in views

$$\boldsymbol{R} \sim N(\bar{\boldsymbol{\mu}}_{\boldsymbol{R}}^{BL}, \bar{\boldsymbol{\sigma}}_{\boldsymbol{R}}^{2BL}) \quad \underset{\boldsymbol{\sigma}^{2view} \to \infty}{\to} \quad \boldsymbol{R} \sim N(\underline{\boldsymbol{\mu}}_{\boldsymbol{R}}, (1+\frac{1}{\tau})\underline{\boldsymbol{\sigma}}_{\boldsymbol{R}}^{2})$$

Limit cases

- High confidence in prior

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- Low confidence in views

$$\boldsymbol{R} \sim N(\bar{\boldsymbol{\mu}}_{\boldsymbol{R}}^{BL}, \bar{\boldsymbol{\sigma}}_{\boldsymbol{R}}^{2BL}) \quad \underset{\boldsymbol{\sigma}^{2view} \to \infty}{\rightarrow} \quad \boldsymbol{R} \sim N(\underline{\boldsymbol{\mu}}_{\boldsymbol{R}}, (1 + \frac{1}{\tau})\underline{\boldsymbol{\sigma}}_{\boldsymbol{R}}^2)$$

- Full confidence in views

$$\boldsymbol{R} \sim N(\bar{\boldsymbol{\mu}}_{\boldsymbol{R}}^{BL}, \bar{\boldsymbol{\sigma}}_{\boldsymbol{R}}^{2BL}) \quad \underset{\boldsymbol{\sigma}^{2view} \to \boldsymbol{0}}{\rightarrow} \quad \boldsymbol{R} \sim N(\bar{\boldsymbol{\mu}}_{\boldsymbol{R}}^{sureBL}, \bar{\boldsymbol{\sigma}}_{\boldsymbol{R}}^{2sureBL})$$

Limit cases

- High confidence in prior

$$\boldsymbol{R} \sim N(\bar{\boldsymbol{\mu}}_{\boldsymbol{R}}^{BL}, \bar{\boldsymbol{\sigma}}_{\boldsymbol{R}}^{2BL}) \quad \underset{\tau \to \infty}{\to} \quad \boldsymbol{R} \sim N(\underline{\boldsymbol{\mu}}_{\boldsymbol{R}}, \underline{\boldsymbol{\sigma}}_{\boldsymbol{R}}^{2})$$

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$$\equiv \underline{\boldsymbol{\mu}}_{\boldsymbol{R}} + \underline{\boldsymbol{\sigma}}_{\boldsymbol{R}}^{2} \boldsymbol{v}_{\mu}' (\boldsymbol{v}_{\mu} \underline{\boldsymbol{\sigma}}_{\boldsymbol{R}}^{2} \boldsymbol{v}_{\mu}')^{-1} (\boldsymbol{\mu}^{view} - \boldsymbol{v}_{\mu} \underline{\boldsymbol{\mu}}_{\boldsymbol{R}})$$

Limit cases

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$$\equiv (1+\frac{1}{\tau})\underline{\boldsymbol{\sigma}}_{\boldsymbol{R}}^{2} - \frac{1}{\tau}\underline{\boldsymbol{\sigma}}_{\boldsymbol{R}}^{2}\boldsymbol{v}_{\mu}'(\boldsymbol{v}_{\mu}\underline{\boldsymbol{\sigma}}_{\boldsymbol{R}}^{2}\boldsymbol{v}_{\mu}')^{-1}\boldsymbol{v}_{\mu}\underline{\boldsymbol{\sigma}}_{\boldsymbol{R}}^{2} \approx \underline{\boldsymbol{\sigma}}_{\boldsymbol{R}}^{2}$$

Three stocks from the S&P 500

Relationship with regression

$$\boldsymbol{R} = \boldsymbol{\beta}\underset{\substack{\downarrow \\ = \boldsymbol{v}_{\mu}\boldsymbol{R}}}{\boldsymbol{Z}} + \boldsymbol{U}$$

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$$\downarrow$$

$$= \boldsymbol{v}_{\mu}\boldsymbol{R}$$

$$\Downarrow$$

$$\boldsymbol{R}|(\boldsymbol{v}_{\boldsymbol{\mu}}\boldsymbol{R} = \boldsymbol{\mu}^{view}) \sim N(\bar{\boldsymbol{\mu}}_{\boldsymbol{R}}^{point}, \bar{\boldsymbol{\sigma}}_{\boldsymbol{R}}^{2point})$$

Relationship with regression

$$\boldsymbol{R} = \boldsymbol{\beta}\boldsymbol{Z} + \boldsymbol{U}$$

$$\downarrow$$

$$= \boldsymbol{v}_\mu \boldsymbol{R}$$

$$\Downarrow$$

$$\boldsymbol{R} | (\boldsymbol{v}_{\boldsymbol{\mu}} \boldsymbol{R} = \boldsymbol{\mu}^{view}) \sim N(\bar{\boldsymbol{\mu}}_{\boldsymbol{R}}^{point}, \bar{\boldsymbol{\sigma}}_{\boldsymbol{R}}^{2point})$$

$$\downarrow$$

$$= \underline{\boldsymbol{\mu}}_{\boldsymbol{R}} + \underline{\boldsymbol{\sigma}}_{\boldsymbol{R}}^2 \boldsymbol{v}_{\boldsymbol{\mu}}' (\boldsymbol{v}_{\boldsymbol{\mu}} \underline{\boldsymbol{\sigma}}_{\boldsymbol{R}}^2 \boldsymbol{v}_{\boldsymbol{\mu}}')^{-1} (\boldsymbol{\mu}^{view} - \boldsymbol{v}_{\boldsymbol{\mu}} \underline{\boldsymbol{\mu}}_{\boldsymbol{R}})$$

$$= \bar{\boldsymbol{\mu}}_{\boldsymbol{R}}^{sureBL}$$

Relationship with regression

$$\begin{tikzcd}[row sep=small] \boldsymbol{R} = \boldsymbol{\beta}\boldsymbol{Z} + \boldsymbol{U} \arrow[d] \\ = \boldsymbol{v}_{\mu}\boldsymbol{R} \end{tikzcd}$$

$$\Downarrow$$

$$\begin{tikzcd} \boldsymbol{R}|(\boldsymbol{v}_{\boldsymbol{\mu}}\boldsymbol{R} = \boldsymbol{\mu}^{view}) \sim N(\bar{\boldsymbol{\mu}}_{\boldsymbol{R}}^{point}, \bar{\boldsymbol{\sigma}}_{\boldsymbol{R}}^{2point}) \arrow[d] \\ = \underline{\boldsymbol{\sigma}}_{\boldsymbol{R}}^{2} - \underline{\boldsymbol{\sigma}}_{\boldsymbol{R}}^{2}\boldsymbol{v}_{\mu}'(\boldsymbol{v}_{\boldsymbol{\mu}}\underline{\boldsymbol{\sigma}}_{\boldsymbol{R}}^{2}\boldsymbol{v}_{\boldsymbol{\mu}}')^{-1}\boldsymbol{v}_{\boldsymbol{\mu}}'\underline{\boldsymbol{\sigma}}_{\boldsymbol{R}}^{2} \\ \neq \bar{\boldsymbol{\sigma}}_{\boldsymbol{R}}^{2sureBL} \end{tikzcd}$$

Relationship with partial views

Minimum relative entropy

$$\bar{f}_{\boldsymbol{R}}(\cdot) \equiv \underset{f\in\mathcal{V}}{\operatorname{argmin}}\, \mathcal{E}(f\|f^{N}_{\underline{\boldsymbol{\mu}}_{\boldsymbol{R}},\underline{\boldsymbol{\sigma}}^2_{\boldsymbol{R}}})$$

$$\downarrow$$

$$\text{s.t. } \mathbb{E}^{f}\{\boldsymbol{v}_{\boldsymbol{\mu}}\boldsymbol{R}\} = \boldsymbol{\mu}^{view}$$

Relationship with partial views

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$$\bar{f}_{\boldsymbol{R}}(\cdot) \equiv \operatorname*{argmin}_{f \in \mathcal{V}} \mathcal{E}(f \| f^{N}_{\underline{\boldsymbol{\mu}}_{\boldsymbol{R}}, \underline{\boldsymbol{\sigma}}^{2}_{\boldsymbol{R}}})$$

$$\downarrow$$

$$\text{s.t. } \mathbb{E}^{f}\{\boldsymbol{v}_{\boldsymbol{\mu}}\boldsymbol{R}\} = \boldsymbol{\mu}^{view}$$

$$\Downarrow \text{Ⓔ}$$

$$\boldsymbol{R}|(\mathbb{E}\{\boldsymbol{v}_{\boldsymbol{\mu}}\boldsymbol{R}\} = \boldsymbol{\mu}^{view}) \sim \bar{f}_{\boldsymbol{R}} \sim N(\bar{\boldsymbol{\mu}}^{MRE}_{\boldsymbol{R}}, \bar{\boldsymbol{\sigma}}^{2MRE}_{\boldsymbol{R}})$$

Relationship with partial views

Minimum relative entropy

$$\bar{f}_{\boldsymbol{R}}(\cdot) \equiv \underset{f \in \mathcal{V}}{\operatorname{argmin}} \, \mathcal{E}(f \| f^{N}_{\underline{\boldsymbol{\mu}}_{\boldsymbol{R}}, \underline{\boldsymbol{\sigma}}^{2}_{\boldsymbol{R}}})$$

$$\downarrow$$

$$\text{s.t. } \mathbb{E}^{f}\{\boldsymbol{v}_{\boldsymbol{\mu}}\boldsymbol{R}\} = \boldsymbol{\mu}^{view}$$

$$\Downarrow \text{Ⓔ}$$

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$$\downarrow$$

$$= \underline{\boldsymbol{\mu}}_{\boldsymbol{R}} + \underline{\boldsymbol{\sigma}}^{2}_{\boldsymbol{R}}\boldsymbol{v}'_{\mu}(\boldsymbol{v}_{\mu}\underline{\boldsymbol{\sigma}}^{2}_{\boldsymbol{R}}\boldsymbol{v}'_{\mu})^{-1}(\boldsymbol{\mu}^{view} - \boldsymbol{v}_{\mu}\underline{\boldsymbol{\mu}}_{\boldsymbol{R}})$$

Relationship with partial views

Minimum relative entropy

$$\bar{f}_{\boldsymbol{R}}(\cdot) \equiv \underset{f \in \mathcal{V}}{\operatorname{argmin}} \, \mathcal{E}(f \| f^{N}_{\underline{\boldsymbol{\mu}}_{\boldsymbol{R}}, \underline{\boldsymbol{\sigma}}^2_{\boldsymbol{R}}})$$

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$$\text{s.t. } \mathbb{E}^{f}\{\boldsymbol{v}_{\boldsymbol{\mu}} \boldsymbol{R}\} = \boldsymbol{\mu}^{view}$$

$$\Downarrow \text{Ⓔ}$$

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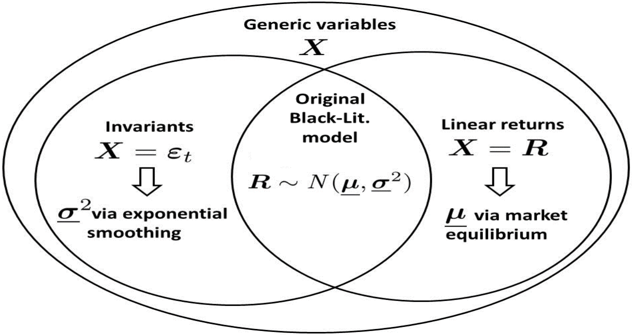
$$\downarrow$$

$$= \underline{\boldsymbol{\sigma}}^2_{\boldsymbol{R}}$$

Three stocks from the S&P 500

Black-Litterman original model on linear returns



Generalizations

From linear returns to risk drivers

From linear returns to risk drivers

From stock-like to generic asset classes

From normal to non-normal markets

From linear equality views to partial flexible views