Interest rate modeling with overnight rates and expected jumps

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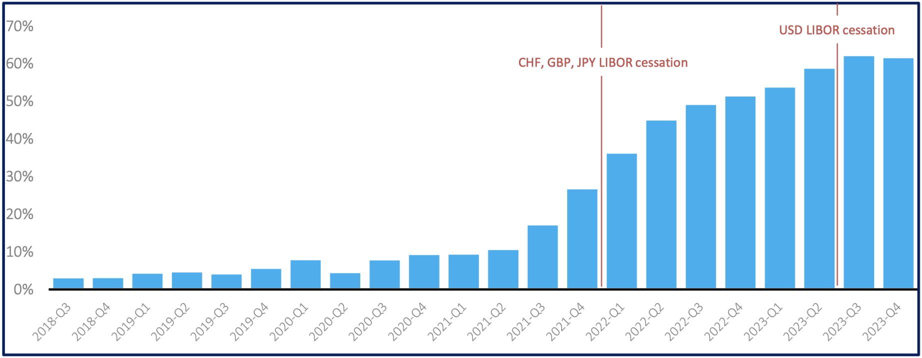
The LIBOR reform

- London Interbank Offered Rate (LIBOR), computed as the trimmed average of rates reported by a panel of banks, for five currencies (CHF, EUR, GBP, JPY, USD) and seven tenors (1D, 1W, 1M, 2M, 3M, 6M, 1Y).
- Since the global financial crisis, the volume of uncollateralized loans in the interbank market shrinked significantly, mainly because of counterparty risk.
- 2012: evidence of LIBOR manipulation by several major banks.
- July 2017: Andrew Bailey (FCA) spoke about *"the future of LIBOR"*: LIBOR discontinuation after 2021.
- March 2021: FCA announced complete LIBOR cessation after June 2023.
- Transition towards transaction-based overnight rates as benchmark rates: SOFR (US), SONIA (UK), TONAR (JP), SARON (CH), €STR (EU).

Adoption of overnight rates



Source: ISDA-Clarus adoption indicator, published on 24/01/2024.

Overnight rates and expected jumps

- The new benchmark rates are risk-free overnight rates (RFRs);
- Being risk-free, RFRs are aligned to policy rates:
  as documented by Backwell and Hayes (2022), most of the variation in SONIA over the years 2016-2020 occurs in correspondence to the meeting dates of the Monetary Policy Committee of the Bank of England.
  $\Rightarrow$ meeting dates follow a predetermined calendar.
- Upward/downward spikes at regulatory reporting dates:
  SOFR was on average 20.25 bps higher at quarter-ends compared to other dates (source: Klingler and Syrstad (2021), period: 08/2014 - 12/2019).

These facts provide empirical evidence of **expected jumps**:
new information that affects interest rates arriving at dates known ex-ante.

SOFR: spikes and hikes

SOFR time series from 01/01/2018 until 12/12/2022 (source: Refinitiv).

SOFR: spikes and hikes

- Let us consider the spike observed on 17/09/2019.
  According to Anbil et al. (2020):

  > *Strains in money markets in September seem to have originated from routine market events, including a corporate tax payment date and Treasury coupon settlement. The outsized and unexpected moves in money market rates were amplified by a number of factors.*

- This analysis suggests that the date of the spike was known in advance, while the size of the jump was not predictable.

- Presence of expected jumps in interest rate models.
  This phenomenon is playing an important role in recent works:
  - Kim and Wright (2014): short rate model with jumps at fixed times.
  - Andersen and Bang (2020): spikes in the SOFR dynamics, both at expected and unexpected times.
  - Gellert and Schlögl (2021): diffusive HJM model for instantaneous forward rates, with jumps/spikes at fixed times in the short rate.
  - Brace et al. (2022): diffusive HJM model with stochastic volatility.
  - Backwell and Hayes (2022): short-rate model for the SONIA rate, based on a pure jump process with expected and unexpected jump times.
  - Schlögl et al. (2023): joint model for policy and overnight benchmark rates.

Outline of the talk

- A Heath-Jarrow-Morton framework with expected jumps;
- The affine setting and some model examples;
- Hedging in the presence of expected jumps.

For more information:

C. Fontana, Z. Grbac, T. Schmidt (2024), Term structure modelling with overnight rates beyond stochastic continuity, *Mathematical Finance*, 34(1): 151–189.

Interest rates and ZCB prices

- The overnight rate r_{t_n} between day t_n and t_{n+1} is

$$r_{t_n} = \frac{1}{\Delta_n}\left(\frac{1}{P(t_n, t_{n+1})} - 1\right),$$

  with $P(t_n, t_{n+1})$ the zero-coupon bond (ZCB) price at t_n for maturity t_{n+1}.
- LIBOR rates are term rates: how to use RFRs to replace them?
- Setting-in-arrears rate:

$$R(S, T) = \frac{1}{T - S}\left(\prod_{n \in N(S,T)} \left(1 + \Delta_n r_{t_n}\right) - 1\right),$$

  where $N(S, T) := \{n \in \mathbb{N} : S \leq t_n < t_{n+1} \leq T\}$.
- According to the ISDA protocol, $R(S, T)$ is adopted as the LIBOR fallback, up to an additive spread determined from historical data.
- This rate is backward-looking since its value is known only at T.
- Forward-looking rate $F(S, T)$: rate K such that the single-period swap delivering $R(S, T) - K$ at maturity T has zero value at time S, so that

$$F(S, T) = \frac{1}{T - S}\left(\frac{1}{P(S, T)} - 1\right).$$

ZCB prices constitute the fundamental basis of the market.

Continuous-time modeling

- For convenience, we model the overnight rate in continuous time:

$$\boldsymbol{\rho_t} = \lim_{\Delta \to 0} r_t = \lim_{\Delta \to 0} \frac{1}{\Delta}\left(\frac{1}{P(t, t+\Delta)} - 1\right) = -\partial_T \ln P(t, T)\Big|_{T=t}.$$

  We call ρ_t the risk-free rate (RFR).
- For all $T \geq t$, the instantaneous forward rate $f(t, T)$ is defined as

$$f(t, T) = -\partial_T \ln P(t, T),$$

  and hence we have that $f(t, t) = \rho_t$.
- ZCB prices are obtained by integration, using the fact that $P(T, T) = 1$:

$$P(t, T) = \exp\left(-\int_t^T f(t, u)du\right).$$

- These computations represent the starting point of the Heath-Jarrow-Morton (1992) framework for interest rate modeling.

An extended HJM framework

We specify ZCB prices as

$$P(t,T) = \exp\left(-\int_t^T f(t,u)du\right),$$

and assume that

$$df(t,T) = \alpha(t,T)dt + \varphi(t,T)dW_t + \Delta V(t,T)\delta_{\mathcal{S}}(t),$$

where

- W_t a d-dim. Brownian motion,
- $\delta_{\mathcal{S}}(t) = 1$ if and only if $t \in \mathcal{S}$,
- $\mathcal{S} = \{s_1, \dots, s_M\}$ is a fixed set of dates.

The set $\mathcal{S}$ contains the expected jump dates: the calendar of dates at which forward rates and the RFR are expected to jump.

Remarks:

- ZCB prices can be generalized with discontinuities in T;
- Lévy-type jumps (unexpected jumps) can be added to the model;
- $\mathcal{S}$ can be generalized to a countable family of times that are announced.

An extended HJM framework

The considered financial market has the following features:

- uncountably many assets (ZCBs for all possible maturities);
- jumps at fixed times.

How can we ensure absence of arbitrage?

Characterize when the probability measure ***Q* is a risk-neutral measure**.

- This corresponds to the (local) martingale property under Q of ZCB prices, discounted with respect to the money market account $\exp(\int_0^{\cdot} \rho_t dt)$.
- Sufficient to ensure *no asymptotic free lunch with vanishing risk* (NAFLVR).

An extended HJM framework

Let us define

$$\bar{\alpha}(t,T) := \int_t^T \alpha(t,u)du, \quad \bar{\varphi}(t,T) := \int_t^T \varphi(t,u)du, \quad \bar{V}(t,T) := \int_t^T \Delta V(t,u)du.$$

Theorem

Q is a risk-neutral measure if and only if the following two conditions are satisfied:

(i) for every $T > 0$ and $t \in [0,T]$, it holds that

$$\bar{\alpha}(t,T) = \frac{1}{2}\|\bar{\varphi}(t,T)\|^2,$$

(ii) for every $T > 0$ and $i = 1,\dots,M$, it holds that

$$E^Q\left[e^{-\bar{V}(s_i,T)}\middle|\mathcal{F}_{s_i-}\right] = 1.$$

Interpretation:

1. drift condition: under Q the rate of return of ZCBs coincides with the RFR;
2. jump condition: impossibility to forecast the magnitude of expected jumps.

Example: a Cheyette-type model

The Cheyette (2001) model is a widely adopted finite-dimensional HJM model.
How can we extend this model to include expected jumps?

- instantaneous forward rates:

$$df(t,T) = \alpha(t,T)dt + \varphi(t,T)dW_t + \big(\alpha_i(T) + \xi_i g_i(T)\big)\delta_{\mathcal{S}}(t),$$

  with independent $\xi_i \sim \mathcal{N}(\mu_i, \sigma_i^2)$, for $i = 1, \dots, M$;
- separable volatility structure (1-factor, for illustration):

$$\varphi(t,T) = \frac{a(T)}{a(t)} b(t) \qquad \text{and} \qquad g_i(T) = a(T)B_i.$$

- Condition (ii) of the above theorem implies that

$$\alpha_i(T) = a(T)B_i \left(\sigma_i^2 B_i \int_{s_i}^{T} a(u)du - \mu_i \right).$$

- It holds that

$$f(t,T) = f(0,T) + \frac{a(T)}{a(t)} X_t + U(t,T),$$

  where X is a mean-reverting Gaussian Markov process with mean-reversion speed $\partial_t \log(a(t))$, diffusion coefficient b and jumps at dates $\{s_1, \dots, s_M\}$.

The affine setting

The presence of expected jump times requires an extension of affine processes: affine semimartingales generalize affine processes by allowing for jumps at fixed times with possibly state-dependent jump sizes (see Keller-Ressel et al. (2019)).

An affine semimartingale $X = (X_t)_{t \geq 0}$ taking values in $\mathbb{R}^m_+ \times \mathbb{R}^n$ satisfies

$$E\left[e^{\langle u, X_T \rangle} | \mathcal{F}_t\right] = \exp\bigl(\phi_t(T,u) + \langle \psi_t(T,u), X_t \rangle\bigr),$$

for all $u \in \mathcal{U} = \mathbb{C}^m_- \times \mathrm{i}\mathbb{R}^n$, where the functions $\phi_t(T,u)$ and $\psi_t(T,u)$ satisfy generalized Riccati equations.

$\Rightarrow$ Short-rate approach: let the RFR be given by

$$\rho_t = \ell(t) + \langle \Lambda, X_t \rangle, \qquad \text{for all } t \geq 0,$$

where the function ℓ fits the initially observed term structure.

Proposition

The joint process $(X_t, \int_0^t \rho_u du)$ is an affine semimartingale.

- Similar to the enlargement of the state-space approach of Duffie et al. (2003).
- Fourier-based methods for pricing a variety of interest rate derivatives.

An example: a two-factor Hull-White model

The Hull-White model as a market standard for RFR modeling. In this example, we propose an extension of a two-factor model ($\Rightarrow$ Vincenzo's talk today).

Let

$$dX^i_t = -a_i X^i_t dt + \sigma_i dW^i_t + dJ^i_t, \qquad \text{for } i = 1, 2,$$

with $\mathrm{Corr}(W^1_t, W^2_t) = \rho$ and independent pure jump processes J^1 and J^2:

$$J^i_t = \sum_{m=1}^{M} \xi^i_m \mathbf{1}_{\{s_m \leq t\}}.$$

- J^1_t (with small a_1) models structural jumps, such as monetary policy changes;
- J^2_t (with large a_2) models spikes, such as liquidity squeezes.

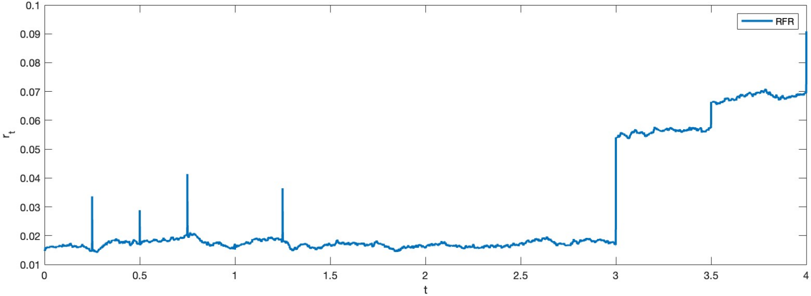
We construct the RFR process ρ_t as

$$\rho_t = \ell(t) + X^1_t + X^2_t.$$

In the Gaussian case (jump sizes ξ^i_m independent and Gaussian):

- explicit formula for ZCB prices;
- Black-type formula for caplets/floorlets on backward-looking rates.

An example: a two-factor Hull-White model



Simulation of the extended two-factor Hull-White model. Model parameters: $\rho = 0.5$ and

	$i = 1$	$i = 2$
a_i	0.1	0.5
σ_i	0.005	0.005
ξ^i	$\mathcal{N}(0.01, 0.05)$	$\mathcal{N}(0.01, 0.01)$

Hedging expected jumps

- For simplicity, consider a single traded asset with price process

$$dS_t = \mu_t dt + \sigma_t dW_t + dJ_t.$$

For example, S can represent the price process of a SOFR futures contract.

- We assume that the only sources of randomness are W and J.
- We want to hedge a derivative with payoff H at maturity T.

If no jumps were present:

- complete market (unique risk-neutral measure $\widehat{Q}$);
- unique arbitrage-free price $\widehat{H}_t = E^{\widehat{Q}}[H|\mathcal{F}_t]$;
- for every payoff H, existence of a replicating strategy Δ^H (Delta hedging):

$$\Delta_t^H = \frac{d\langle \widehat{H}, S\rangle_t}{d\langle S\rangle_t} = \frac{\text{“ } \mathrm{Cov}(\widehat{H}, S) \text{ ”}}{\text{“ } \mathrm{Var}(S) \text{ ”}}.$$

Hedging expected jumps

If jumps are present:

- Expected jumps induce market incompleteness:
  we know when a jump is going to occur, but we cannot foresee its impact.
- We therefore make use of the concept of local risk-minimization:
  perfect replication, but at a cost.

In the presence of expected jumps, the optimal hedging strategy θ^H is given by

$$\theta_t^H = \frac{d\langle \widehat{H}, S\rangle_t}{d\langle S\rangle_t} = \Delta_t^H + \frac{\mathrm{Cov}(\Delta \widehat{H}_t, \Delta S_t | \mathcal{F}_{t-})}{\mathrm{Var}(\Delta S_t | \mathcal{F}_{t-})},$$

where $\widehat{H}_t = E^{\widehat{Q}}[H|\mathcal{F}_t]$ and $\widehat{Q}$ denotes the minimal martingale measure.

Remarks:

- The first component Δ^H provides perfect replication between jump dates;
- the second component is a conditional regression coefficient at jump dates;
- the residuals of the regression are the cost to achieve perfect replication.

Conclusions and outlook

- Expected jumps as an essential feature of interest rate markets;
- Heath-Jarrow-Morton framework ensuring absence of arbitrage;
- tractable models for pricing applications and hedging;
- many open questions:
  - estimation and practical implementation;
  - transition from risk-neutral to real-world;
  - regime switching at known dates.

Thank you for your attention

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